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RECIPROCITY IN A PROBLEM OF RELATIVE MAXIMA AND MINIMA.*

By James K. Whittemore.

Introduction.—Most students of the differential calculus have doubtless observed that the two following problems have the same solution: To determine the shape of a rectangle of given perimeter and *maximum* area; To determine the shape of a rectangle of given area and *minimum* perimeter. While in this and similar cases a quite elementary explanation may be given of the exchange of maximum and minimum corresponding to the exchange of area and perimeter it seems of interest to consider the analytical problem of maximum or minimum value of a function of two variables, subject to the condition that a second function of the two variables have a constant value, and to determine when the exchange of the rôles of the two functions results in the exchange of maximum and minimum. In this paper this question of reciprocity is discussed for the simplest problem, two functions of two variables. It is suggested that a similar discussion of the more general problem with more than two variables and with two or more than two functions might give results of considerable interest. The methods are analytical but are based largely on geometrical intuition. It is intended that the discussion should be complete, that is that all necessary and sufficient conditions should be given, in so far as this is possible with the use of the first and second partial derivatives of the two functions. In the first section the general case is considered, simple necessary and sufficient conditions for a relative extreme, maximum or minimum, obtained and stated in both analytical and geometrical form, and the reciprocity condition determined. In the second and third sections the more complicated exceptional cases are discussed. The fourth section is devoted to a study of the invariant properties of the conditions previously established. In the last section we give examples illustrating the theory developed in the preceding sections.

§ 1. **The General Case.**—Consider two functions φ and ψ of the two real variables, x and y; suppose that in the neighborhood of $P(x_0, y_0)$ both φ and ψ are single valued, real, and have continuous partial derivatives of the first and second orders. We write

$$\varphi(x, y) = u, \qquad \psi(x, y) = v,$$

* Presented to the American Mathematical Society at New York, April 24, 1920.

and denote the values of these functions at P by u_0 and v_0. We consider the two problems:

A. To determine when u_0 is an extreme of φ subject to the condition, $\psi = v_0$.

B. To determine when v_0 is an extreme of ψ subject to the condition, $\varphi = u_0$.

It is necessary to specify exactly what is meant by the statement, u_0 is, for example, a maximum of φ subject to the condition $\psi = v_0$. This statement shall in the following pages be understood to mean: (1) There exists a continuous set of real values x, y, of which set x_0, y_0 is an interior point, and of which all points satisfy the equation $\psi = v_0$; (2) For every point of this set, different from x_0, y_0, $u_0 > \varphi(x, y)$. Consider with this definition of relative maximum the example

$$\varphi = y^2, \qquad \psi = x^2 + y^2,$$
$$x_0 = y_0 = u_0 = v_0 = 0.$$

The condition, $x^2 + y^2 = 0$, gives $\varphi = -x^2$, and 0 is a maximum value of φ considered as a function of the independent real variable x, but according to the definition above is not a maximum relative to the condition, $x^2 + y^2 = 0$, for there is not a continuous set of *real* values, x, y, including 0, 0, satisfying the equation of condition. In some circumstances, as will appear in § 2, there are two or more sets of real values x, y satisfying the condition.* It may happen that u_0 is a maximum on both sets, supposing that there are two such sets, a maximum on one and not on the second, or not a maximum on either.

Evidently problems A and B are interchanged by exchanging φ and ψ. Problem A may be stated in geometrical form as follows: We regard x and y as ordinary rectangular coördinates in a plane which we find it convenient to call horizontal; the problem is that of an extreme of the ordinate u of the curve of intersection of the surface, S, $\varphi = u$, with the cylinder, $\psi = v_0$. Suppose first that P is not a singular point of the curve C in the xy plane, $\psi = v_0$; then one branch of C passes through P, and there is no essential restriction† in supposing that, for P, $\psi_y \neq 0$. Evidently a necessary condition for an extreme in A is that the tangent to the curve of intersection of S and cylinder at Q (x_0, y_0, u_0) be horizontal. For this curve

$$\frac{du}{dx} = \varphi_x + y'\varphi_y = 0, \qquad \frac{dv_0}{dx} = \psi_x + y'\psi_y = 0, \qquad y' = \frac{dy}{dx} = -\frac{\psi_x}{\psi_y}.$$

* Usually in the following pages we consider only such a set of points (x, y) as form a curve through P having at P a continuously turning tangent.

† See § 4.

It is necessary that, at P,

$$J = \varphi_x \psi_y - \psi_x \varphi_y = 0.$$

The "critical points," such as x_0, y_0, are determined in the usual elementary solution of A by solving simultaneously $\psi = v_0$ and $J = 0$.

For P we have

$$\frac{d^2u}{dx^2} = y'' \varphi_y + y'^2 \varphi_{yy} + 2y' \varphi_{xy} + \varphi_{xx}, \qquad y'' = \frac{d^2y}{dx^2},$$

$$\frac{d^2v_0}{dx^2} = y'' \psi_y + y'^2 \psi_{yy} + 2y' \psi_{xy} + \psi_{xx} = 0.$$

If, at P, $\varphi_y = 0$ it follows from $J = 0$ that $\varphi_x = 0$. This case we consider in § 2, supposing now $\varphi_y \neq 0$. We have at P

$$\frac{d^2u}{dx^2} = \frac{\psi_x^2}{\psi_y^2} \varphi_{yy} - 2\frac{\psi_x}{\psi_y} \varphi_{xy} + \varphi_{xx} - \frac{\varphi_y}{\psi_y}\left(\frac{\psi_x^2}{\psi_y^2}\psi_{yy} - 2\frac{\psi_x}{\psi_y}\psi_{xy} + \psi_{xx}\right)$$

$$= \varphi_y \left[\frac{\varphi_x^2 \varphi_{yy} - 2\varphi_x \varphi_y \varphi_{xy} + \varphi_y^2 \varphi_{xx}}{\varphi_y^3} - \frac{\psi_x^2 \psi_{yy} - 2\psi_x \psi_y \psi_{xy} + \psi_y^2 \psi_{xx}}{\psi_y^3}\right],$$

since $J = 0$. We may write for P

$$\frac{d^2u}{dx^2} = \varphi_y (y_\psi'' - y_\varphi''),$$

where y_φ'' and y_ψ'' are the values at P of d^2y/dx^2 for the curves in the xy plane, $\varphi = u_0$ and $\psi = v_0$ respectively. For problem B we should evidently have at P the necessary condition $J = 0$, and

$$\frac{d^2v}{dx^2} = \psi_y (y_\varphi'' - y_\psi'').$$

We may state the following: With the hypothesis that each of the real curves in the xy plane, $\varphi = u_0$ and $\psi = v_0$, has an ordinary point at P, a condition necessary in both A and B is that these curves be tangent at P, $J = 0$; if this is the case a sufficient condition in both A and B is that the curves do not osculate at P; if both the necessary and sufficient conditions are satisfied A and B have like or unlike extremes as φ_y and ψ_y have different signs or the same sign at P. If the two curves osculate at P, so that the sufficient condition fails, problems A and B cannot be completely discussed by use of the first and second derivatives; we shall say in such a case the discussion fails.*

§ 2. **Exceptional Cases.**—We consider in this section the cases of problem A excluded in the preceding discussion. The conditions stated in each

* For example see § 5.

case are for the values x_0, y_0.

I. $\psi_x = \psi_y = 0$, φ_x and φ_y not both zero.

II. $\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero.

III. $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

To put the problem in geometrical form we consider x, y as before to be rectangular coördinates in a horizontal plane, and consider the intersection of the surface S with the cylinder, $\psi = v_0$, a cylinder with vertical elements erected on the curve, C, $\psi = v_0$, in the xy plane. This curve has in I and III a singular point at P. We assume that not all three of the second partial derivatives of ψ vanish at P, so that this point is a double point. This point is an isolated point with imaginary tangents, a cusp or osculating point or isolated point with one real tangent, or a point of intersection of two real branches of the curve with distinct real tangents, as $\Delta\psi = \psi_{xy}^2 - \psi_{xx}\psi_{yy}$ is negative, zero, or positive at P. The slopes of the tangents are the values of y' satisfying

$$y'^2\psi_{yy} + 2y'\psi_{xy} + \psi_{xx} = 0.$$

In II and III the tangent plane to S is horizontal at Q. The total curvature of the surface at this point is positive, zero, or negative, for all cases, as $\Delta\varphi$ is negative, zero, or positive, and the slopes of the horizontal projections of the asymptotic tangents at the point are in all cases given by the values of y' satisfying

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = 0.$$

In accordance with the limitations set for our discussion we suppose that not all of the second partial derivatives of φ vanish at Q, or, in geometrical terms, that Q is not a flat point of the surface S.

Case I.—$\psi_x = \psi_y = 0$, φ_x and φ_y not both zero. P is a double point of C; the tangent plane to the surface S at Q is not horizontal, but contains a single horizontal direction given by $y'\varphi_y + \varphi_x = 0$. We assume, as in the first section, $\varphi_y \neq 0$.

I*a*. $\Delta\psi < 0$. No real branch of C passes through P, and u_0 is not an extreme of φ.

I*b*. $\Delta\psi = 0$. Suppose the point P is a cusp or an osculating point of C. If P is a cusp and if the cuspidal tangent is not parallel to the horizontal tangent to S, at Q, that is if

$$D_\psi\varphi = \varphi_x^2\psi_{yy} - 2\varphi_x\varphi_y\psi_{xy} + \varphi_y^2\psi_{xx} \neq 0,$$

the ordinate u of a point moving on the curve of intersection of surface and cylinder will increase to u_0 then decrease, or decrease to u_0 then increase, since $du/dx \neq 0$ at P, and u_0 is an extreme of φ. The discussion fails in this case to distinguish between maximum and minimum. If P is an osculating

point of C and if a path with no cusp at P be followed a necessary condition for an extreme is $D_{\psi}\varphi = 0$. The discussion fails for further conditions, and fails also to distinguish between cusp, osculating point, and isolated point.

I*c*. $\Delta\psi > 0$. The point P is a double point of C with two real distinct tangents. In this case we consider only a path with a continuously turning tangent, that is a path projected into a branch of C. Evidently, for one branch of C, u_0 will not be an extreme of φ. In order that u_0 shall be an extreme for one branch of C it is necessary that $D_{\psi}\varphi = 0$. Further discussion of this case fails.

Case II.—$\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero. The point P is a simple point of C; the tangent plane to S at Q is horizontal. In all cases under II, since, at P, y_{ψ}'' is finite with the assumption $\psi_y \neq 0$,

$$\frac{d^2u}{dx^2} = \frac{D_{\phi}\psi}{\psi_y^2}.$$

II*a*. $\Delta\varphi < 0$. The total curvature of S is positive at Q, and u_0 is an absolute and consequently a relative extreme of φ, a maximum or minimum as $D_{\phi}\psi \neq 0$ is negative or positive. It is evident that φ_{xx} and φ_{yy} have the same sign as $D_{\phi}\psi$.

II*b*. $\Delta\varphi = 0$. The total curvature of S vanishes at Q, which is a parabolic point, since we have excluded the case of a flat point. There is at Q a single asymptotic tangent given by

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = 0.$$

It is clear geometrically that u_0 is a relative extreme of φ if the tangent to C at P is not parallel to the asymptotic tangent, that is, if $D_{\phi}\psi \neq 0$. As in II*a* the nature of the extreme is given by the sign of $D_{\phi}\psi$, the same as that of φ_{xx} and φ_{yy} when neither of the latter vanishes. If $D_{\phi}\psi = 0$ the discussion fails.

II*c*. $\Delta\varphi > 0$. The total curvature of S is negative at Q, which is accordingly a hyperbolic point. There are at Q two distinct real asymptotic tangents given by the same equation as in II*b*. If the tangent to C at P is not parallel to either asymptotic tangent u_0 is an extreme of φ, whose nature is given as in II*a* and *b* by the sign of $D_{\phi}\psi$, but not by the sign of φ_{xx} or φ_{yy}. If $D_{\phi}\psi = 0$ the tangent at P is parallel to one of the asymptotic tangents and further discussion fails.

Case III.—$\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

The point P is a double point of C; the tangent plane to S at Q is horizontal.

III*a*. $\Delta\varphi < 0$. The total curvature of S is positive at Q.

1. $\Delta\psi < 0$. No real branch of C passes through P, and u_0 is not an extreme of φ.

2. $\Delta\psi = 0$, 3. $\Delta\psi > 0$. Since u_0 is an absolute extreme of φ it is a relative extreme for any path, a maximum or minimum as φ_{yy} is negative or positive, if P is not an isolated point of C in 2.

III*b*. $\Delta\varphi = 0$. The total curvature of S vanishes at Q, and there is a single asymptotic tangent.

1. $\Delta\psi < 0$. There is no extreme of φ at u_0.

2. $\Delta\psi = 0$. The curve C has a single tangent at the double point P. It is evident geometrically that u_0 is an extreme of φ if P is not an isolated point of C and if the tangent at P is not parallel to the asymptotic tangent at Q.

To express the last condition analytically consider the values at $P(x_0, y_0)$ of the two polynomials in y',

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = \varphi_{yy}(y' - \alpha_1)(y' - \alpha_2)$$
$$y'^2\psi_{yy} + 2y'\psi_{xy} + \psi_{xx} = \psi_{yy}(y' - \beta_1)(y' - \beta_2).$$

Their resultant R is

$$R = \varphi_{yy}^2\psi_{yy}^2(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2) = H^2 - 4\Delta\varphi\Delta\psi,$$

where

$$H = \varphi_{xx}\psi_{yy} + \psi_{xx}\varphi_{yy} - 2\varphi_{xy}\psi_{xy}.$$

The condition that the tangents named be not parallel is $R \neq 0$, or since $\Delta\varphi = 0$, $H \neq 0$. As in III*a*, 2 and 3, the nature of the extreme is given by the sign of φ_{yy}, if $\varphi_{yy} \neq 0$; by the sign of φ_{xx} if $\varphi_{yy} = 0$; not both φ_{xx} and φ_{yy} vanish, and they have the same sign if neither vanishes. If $H = 0$ the two tangents are parallel and the discussion fails.

3. $\Delta\psi > 0$. The curve C has two real branches intersecting at P with distinct tangents. It is evident geometrically that u_0 is always an extreme of φ for one branch of C, and an extreme of the same kind for both branches if the tangent to neither branch is parallel to the asymptotic tangent to S at Q. Considered analytically, we have $R = H^2$, since $\Delta\varphi = 0$, and $\alpha_1 = \alpha_2$. If $H \neq 0$, we have for the two branches of C, since y_ψ'' is finite, d^2u/dx^2 equal to $\varphi_{yy}(\beta_1 - \alpha_1)^2$ and $\varphi_{yy}(\beta_2 - \alpha_1)^2$, $\varphi_{yy} \neq 0$; the two extremes are alike, maximum or minimum as φ_{yy} is negative or positive. If $H = 0$ the tangent to one branch of C is parallel to the asymptotic tangent at Q, and for this branch the discussion fails; for the other branch the nature of the extreme is given as before by the sign of φ_{yy}.

III*c*. $\Delta\varphi > 0$. The total curvature of S is negative at Q; there are two distinct asymptotic tangents.

1. $\Delta\psi < 0$. There is no extreme of φ at u_0.

2. $\Delta\psi = 0$. It is evident geometrically that sufficient conditions that u_0 be a relative extreme of φ are that P is not an isolated point of C and that the single tangent to C at P is not parallel to either asymptotic tangent at Q, that is $H \neq 0$. If $H = 0$ the discussion fails.

We determine the nature of the extreme when $H \neq 0$ as follows: In the neighborhood of x_0, y_0, we have

$$\varphi = u_0 + (\Delta y)^2\varphi_{yy} + 2\Delta x\Delta y\varphi_{xy} + (\Delta x)^2\varphi_{xx},$$

for values of the partial derivatives at a point near x_0, y_0. Since the derivatives are continuous by hypothesis u_0 is a maximum or minimum as

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = \varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad y' = \beta_1 = \beta_2,$$

is negative or positive. Now it is easily proved that

$$H = \tfrac{1}{2}\varphi_{yy}\psi_{yy}\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_1)\},$$

and, since in the case before us $\beta_1 = \beta_2$, we have

$$H = \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)(\alpha_2 - \beta_1).$$

Then u_0 is a maximum or minimum as $\psi_{yy}H$ is negative or positive. The value of d^2u/dx^2 cannot be used in this discussion since in general, when $\Delta\psi = 0$, y_ψ'' becomes infinite.

3. $\Delta\psi > 0$. For both branches of C y_ψ'' is finite if $\psi_{yy} \neq 0$, and, if $\varphi_{yy} \neq 0$,

$$\frac{d^2u}{dx^2} = \varphi_{yy}(y' - \alpha_1)(y' - \alpha_2).$$

If $R < 0$, the values of this second derivative for the two branches of C,

$$\varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2),$$

have opposite signs, and u_0 is an extreme of φ of opposite kinds for the two branches of C. If $R > 0$, the two values have the same sign and u_0 is an extreme of φ of the same kind for the two branches of C. If $R = 0$, one or both tangents at P are parallel to respectively one or both asymptotic tangents at Q. Along such a branch of C the discussion fails. If both tangents at P are parallel to asymptotic tangents at Q we must have

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

We consider the nature of the extremes of φ at u_0. Suppose first $R > 0$; there are two extremes of the same kind. If φ_{yy} and ψ_{yy} are both different from zero these extremes are maxima or minima as the following expression,

not zero, is negative or positive:

$$\varphi_{yy}\{(\alpha_1-\beta_1)(\alpha_2-\beta_1)+(\alpha_1-\beta_2)(\alpha_2-\beta_2)\}=\frac{2}{\psi_{yy}^2}U_\phi,$$

$$U_\phi=\psi_{yy}H+2\varphi_{yy}\Delta\psi.$$

The nature of the extremes is then given by the sign of U_ϕ. It will appear in § 4 that the restriction $\varphi_{yy}\psi_{yy}\neq 0$ is not essential. If $R=0$, $\alpha_1=\beta_1$, $\alpha_2\neq\beta_2$, the criterion for maximum or minimum on the branch of C for which $y'=\beta_2$ is again the sign of U_ϕ, unless $\varphi_{yy}=\psi_{yy}=U_\phi=0$, when the distinction will be given by the sign of

$$U_\phi'=\psi_{xx}H+2\varphi_{xx}\Delta\psi\neq 0.$$

We shall show in § 4 that U_ϕ and U_ϕ' have the same sign when neither vanishes.

In three other cases, previously considered, the nature of the extreme or extremes is given by the sign of U_ϕ or of U_ϕ' when the former vanishes. These all come under III, $\varphi_x=\varphi_y=\psi_x=\psi_y=0$; they are b, $\Delta\varphi=0$, 2, $\Delta\psi=0$; b, 3, $\Delta\psi>0$; c, $\Delta\varphi>0$, 2, $\Delta\psi=0$. In the discussion of b, 2, $H\neq 0$, it was shown that u_0 is a relative maximum or minimum as φ_{yy}, supposed different from zero, is negative or positive; in this case $U_\phi=\psi_{yy}H$ and this expression has the same sign as φ_{yy} if ψ_{yy} does not vanish, for

$$\varphi_{yy}\psi_{yy}H=\varphi_{yy}^2\psi_{yy}^2(\alpha_1-\beta_1)^2>0.$$

In b, 3 the criterion was again the sign of φ_{yy}, which has the same sign as U_ϕ, when these are both different from zero, whether or not H vanishes; if either H or ψ_{yy} vanishes $U_\phi=\varphi_{yy}\Delta\psi$; if $\psi_{yy}\neq 0$, we have from the preceding paragraphs,

$$\varphi_{yy}\psi_{yy}^2\{(\alpha_1-\beta_1)^2+(\alpha_1-\beta_2)^2\}=2U_\phi.$$

In c, 2 it was proved that if $H\neq 0$ the extreme is maximum or minimum as $\psi_{yy}H$ is positive or negative; since $\Delta\psi=0$, this is exactly U_ϕ. If $H=0$, $U_\phi=U_\phi'=0$, and the discussion fails. If $\Delta\varphi=\Delta\psi=R=0$ we have the excluded case,

$$\frac{\varphi_{xx}}{\psi_{xx}}=\frac{\varphi_{xy}}{\psi_{xy}}=\frac{\varphi_{yy}}{\psi_{yy}}.$$

If $\Delta\varphi>0$, $\Delta\psi=R=0$ the discussion fails. In all other cases where

$$\Delta\varphi\geqq 0,\qquad \Delta\psi\geqq 0,\qquad R\geqq 0,$$

φ has at u_0 a maximum or a minimum as U_φ is negative or positive; if $U_\phi=0$ the criterion is the sign of $U_\phi'\neq 0$. In certain cases φ has at u_0 two like extremes whose nature is given by the sign of U_ϕ or U_ϕ', namely

$\Delta\varphi \geqq 0$, $\Delta\psi > 0$, $R > 0$. In other cases φ has at u_0 one extreme whose nature is given by the sign of U_ϕ of U_ϕ', and may have a second extreme of either kind or no other extreme, namely $\Delta\varphi \geqq 0$, $\Delta\psi > 0$, $R = 0$.

§ 3. **Reciprocity in the Exceptional Cases.**—We now consider problems A and B together under the same assumed conditions with the purpose of determining, in so far as it is possible with the use of the first and second derivatives of φ and ψ, when extremes in the two problems are like or unlike.

It is clear that under assumed conditions for example in case I of § 2 the solution of B is given by interchanging φ and ψ in II.

In I, $\psi_x = \psi_y = 0$, φ_x and φ_y are not both zero. I*a*, $\Delta\psi < 0$. In A there is no extreme. For B the solution is given by II*a*, exchanging φ and ψ; there is always an extreme.

I*b*. $\Delta\psi = 0$. In A if P is not an isolated point of C a sufficient condition for an extreme is $D_\psi\varphi \neq 0$. For B the solution is that of II*b*; a sufficient condition is $D_\psi\varphi \neq 0$. The discussion fails for comparison of the nature of the extremes in A and B. The sufficient condition common to the two problems is, geometrically expressed, that the two curves, $\varphi = u_0$ and $\psi = v_0$, are not tangent at P.

I*c*. $\Delta\psi > 0$. In A it is necessary for an extreme that $D_\psi\varphi = 0$. For an extreme in B a sufficient condition, by II*c*, is $D_\psi\varphi \neq 0$.

The solutions of A and B with the assumed conditions of case II are given by exchanging φ and ψ in the preceding paragraphs.

In III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

III*a*. $\Delta\varphi < 0$. 1. $\Delta\psi < 0$. There is no extreme in A or B.

2. $\Delta\psi = 0$. There is always an extreme in A unless P is an isolated point of C. The solution of B is given by III*b*, 1; there is no extreme.

3. $\Delta\psi > 0$. There is an extreme for any path in A. In B, by III*c*, 1, there is no extreme.

III*b*. $\Delta\varphi = 0$. 2. $\Delta\psi = 0$. In both A and B, if P is not an isolated point of either curve, $\varphi = u_0$ or C, a sufficient condition for an extreme is $H \neq 0$. We prove that the extremes are like or unlike as H is positive or negative: Since $\Delta\varphi = \Delta\psi = 0$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$; since $R = H^2 \neq 0$, $\alpha_1 \neq \beta_1$ and $H = \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)^2$. Then φ_{yy} and ψ_{yy} have like or unlike signs as H is positive or negative.

3. $\Delta\psi > 0$. In A there is always one extreme, and if $H \neq 0$ two like extremes. In B, if P is not an isolated point of the curve $\varphi = u_0$, a sufficient condition, by III*c*, 2, is $H \neq 0$. If $H \neq 0$ the extremes of A and B are like or unlike as φ_{yy} and $H\varphi_{yy}$ have like or unlike signs, that is as H is positive or negative.

III*b*. $\Delta\varphi > 0$. 3. $\Delta\psi > 0$. If $R < 0$ there are, as has been shown in the preceding section, two unlike extremes in problem A, and similarly two unlike extremes in B.

For $R = 0$, we have, with the notation of § 2, $\alpha_1 \neq \alpha_2$ since $\Delta\varphi > 0$, and $\beta_1 \neq \beta_2$ since $\Delta\psi > 0$, and $\alpha_1 = \beta_1$. If also $\alpha_2 = \beta_2$ then

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}},$$

and further discussion fails in both A and B. If these equations do not hold $\alpha_2 \neq \beta_2$ and there is certainly one extreme for both problems. For A and B respectively, if $\varphi_{yy}\psi_{yy} \neq 0$,

$$\frac{d^2u}{dx^2} = \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2), \qquad \frac{d^2v}{dx^2} = \psi_{yy}(\alpha_2 - \beta_1)(\alpha_2 - \beta_2),$$

and since, for this case,

$$H = \tfrac{1}{2}\varphi_{yy}\psi_{yy}(\alpha_1 - \beta_2)(\alpha_2 - \beta_1),$$

the extremes known to exist in A and B are like or unlike as H is positive or negative.

For $R > 0$ there are two like extremes in A, similarly two like extremes in B. We shall show again that the extremes of A are like or unlike those of B as H is positive or negative. Supposing as before $\varphi_{yy}\psi_{yy} \neq 0$, the two values of d^2u/dx^2 for the two branches of C in A have the same sign and are, as given in § 2,

$$\varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2).$$

The two values of d^2v/dx^2 in B are similarly

$$\psi_{yy}(\alpha_1 - \beta_1)(\alpha_1 - \beta_2), \qquad \psi_{yy}(\alpha_2 - \beta_1)(\alpha_2 - \beta_2)$$

and have the same sign. The signs of the two products,

$$\varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)(\alpha_2 - \beta_2), \qquad \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_2)(\alpha_2 - \beta_1),$$

are alike and the same as the sign of H since the latter is one half their sum. Then the extremes of A and B are like or unlike as H is positive or negative.

We note that in all of the exceptional cases of § 2, where it is possible to determine the nature of the extremes, supposed alike when there are two, in both A and B by the use of the first and second derivatives of φ and ψ, the extremes in the two problems are the same or different as H is positive are negative. These all come under III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$, and include all cases where neither discriminant, $\Delta\varphi$ or $\Delta\psi$, nor the resultant R is negative, not all three vanish, and $H \neq 0$.

§ 4. **Invariant Properties.**—The various analytic conditions derived in the preceding sections must be independent of the geometrical representation of the problem and are accordingly invariant for any change of variables,

not singular at the point considered. It is of interest to consider directly the invariant properties of these conditions. It is also necessary to demonstrate the unessential character of assumptions as to non-vanishing of certain derivatives made in some of the discussions.

Suppose x and y are replaced in φ and ψ by two real functions of two new real variables, x' and y'. Concerning these functions we assume that both have for the point considered finite first and second partial derivatives; further, that at the same point the transformation is not singular, that is

$$\delta = x_{x'}y_{y'} - x_{y'}y_{x'} \neq 0.$$

For the new variables the various conditions will be obtained by replacing the derivatives of φ and ψ with respect to x and y in the conditions established by the corresponding derivatives of the same functions with respect to x' and y'. We have

$$\begin{aligned}
\varphi_{x'} &= \varphi_x x_{x'} + \varphi_y y_{x'}, \qquad \varphi_{y'} = \varphi_x x_{y'} + \varphi_y y_{y'}. \\
\varphi_{x'x'} &= \varphi_{xx} x_{x'}^2 + 2\varphi_{xy} x_{x'} y_{x'} + \varphi_{yy} y_{x'}^2 + \varphi_x x_{x'x'} + \varphi_y y_{x'x'}, \\
\varphi_{x'y'} &= \varphi_{xx} x_{x'} x_{y'} + \varphi_{xy}(x_{x'}y_{y'} + x_{y'}y_{x'}) + \varphi_{yy} y_{x'} y_{y'} + \varphi_x x_{x'y'} + \varphi_y y_{x'y'}, \\
\varphi_{y'y'} &= \varphi_{xx} x_{y'}^2 + 2\varphi_{xy} x_{y'} y_{y'} + \varphi_{yy} y_{y'}^2 + \varphi_x x_{y'y'} + \varphi_y y_{y'y'},
\end{aligned}$$

with similar equations for ψ.

Consider the quadratic form,

$$\begin{aligned}
Ap^2 + 2Bpq + Cq^2 &= A'p'^2 + 2B'p'q' + C'q'^2, \\
p = ap' + bq', &\qquad q = cp' + dq', \\
A' &= Aa^2 + 2Bac + Cc^2, \\
B' &= Aab + B(ad + bc) + Ccd. \\
C' &= Ab^2 + 2Bbd + Cd^2.
\end{aligned}$$

If we write

$$a = x_{x'}, \qquad b = x_{y'}, \qquad c = y_{x'}, \qquad d = y_{y'},$$

we may consider φ_x and φ_y as variables contragredient to p and q.* In the following discussion of the invariant properties of the conditions of the preceding sections it is to be remembered that the values of all derivatives of φ, ψ, x, and y considered are the values at P (x_0, y_0). In the first place the classification of § 1 and in § 2, cases I, II, III is invariant, for, since $\delta \neq 0$, $\varphi_{x'} = \varphi_{y'} = 0$ if and only if $\varphi_x = \varphi_y = 0$. Next, if φ_x and φ_y are not both zero, and if ψ_x and ψ_y are not both zero it is clear that x' and y' may be chosen so that $\varphi_{y'}$ and $\psi_{y'}$ are both different from zero. The necessary condition of § 1, $J = 0$, is invariant for J is a contravariant* of weight one. The condition of the same section, sufficient for an extreme

* Bôcher, "Introduction to Higher Algebra," p. 109.

and determining the nature of the extreme, we consider later. When the sufficient condition is satisfied for both A and B the condition that the extremes are like or unlike is that φ_y and ψ_y have unlike or like signs. If ψ_y and $\psi_{y'}$ are both different from zero this condition is invariant if $J = 0$, for

$$\frac{\varphi_{y'}}{\psi_{y'}} = \frac{b\varphi_x + d\varphi_y}{b\psi_x + d\psi_y} = \frac{\varphi_y}{\psi_y}.$$

We may say that φ_y/ψ_y is an absolute conditional invariant, with condition $J = 0$.

In § 2 the further subdivision of the problem depends on the signs of $\Delta\varphi$ and $\Delta\psi$, $\Delta\varphi = \varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}$. It is evident from the values given above for the second derivatives of φ with respect to x' and y' that $\Delta\varphi$ is not in general invariant, but with the conditions, $\varphi_x = \varphi_y = 0$, the transformation is exactly the transformation of the quadratic form in p and q, and since the discriminant is an invariant of weight two in the algebraic theory,* we have conditionally $\Delta'\varphi = \delta^2\Delta\varphi$. The sign of $\Delta\varphi$ and the vanishing of $\Delta\varphi$, and similarly $\Delta\psi$, are therefore conditional invariants. If $\psi_x = \psi_y = 0$, the expression,

$$D_\psi\varphi = \varphi_x^2\psi_{yy} - 2\varphi_x\varphi_y\psi_{xy} + \varphi_y^2\psi_{xx},$$

is invariant in sign, for it is in the algebraic theory the form adjoint to $\psi_{xx}p^2 + 2\psi_{xy}pq + \psi_{yy}q^2$, and therefore a contravariant of weight two.†

We now consider the invariance of sign of $\varphi_y(y_\psi'' - y_\phi'')$, supposed not zero, in § 1. We may evidently write

$$\varphi_y(y_\psi'' - y_\phi'') = \frac{1}{\psi_y^2}\left(D_\phi\psi - \frac{\varphi_y}{\psi_y}D_\psi\psi\right),$$

where we suppose φ_y and ψ_y both different from zero, and $J = 0$. Under these conditions the sign of the expression is evidently unchanged by transformation if $\varphi_{y'}$ and $\psi_{y'}$ are also both different from zero, since φ_y/ψ_y is an absolute invariant, and since the terms of $D_\phi'\psi$ and $D_\psi'\psi$, which contain the second derivatives of x and y with respect to x' and y', cancel.

In case III we have the conditions, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$, so that the transformation is equivalent algebraically to that of two quadratic forms. We have supposed in that discussion that the following equations do not hold:

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}},$$

* Bôcher, l. c., p. 129.
† Bôcher, l. c., p. 159.

which include the special cases also excluded,

$$\varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0, \qquad \text{or} \qquad \psi_{xx} = \psi_{xy} = \psi_{yy} = 0.$$

That this condition is invariant when the four first derivatives vanish is obvious from the values given above for the second derivatives of φ with respect to x' and y'. From the same expressions it follows that under the assumed conditions it is possible so to choose x' and y' that neither φ_{yy} nor ψ_{yy} vanishes. In the same case we have stated that the nature of the extreme is in certain subcases dependent on the sign of φ_{yy} or ψ_{yy}. We have

$$\varphi_{y'y'} = \varphi_{xx}b^2 + 2\varphi_{xy}bd + \varphi_{yy}d^2,$$

and the signs of φ_{yy} and $\varphi_{y'y'}$ are the same when this form is definite, $\Delta\varphi < 0$, or, if neither vanishes, when the form is singular, $\Delta\varphi = 0$. Still considering the same case we have from the algebraic theory* the facts that H and R are invariants of weight two and four respectively, and are therefore unchanged in sign by transformation.

It remains only to consider the invariance of the sign of

$$U_\phi = \psi_{yy}H + 2\varphi_{yy}\Delta\psi,$$

the criterion for maximum or minimum in III, *c*, 3, $\Delta\varphi > 0$, $\Delta\psi > 0$; $R \geqq 0$. We recall the fact that both H and $\Delta\psi$ are invariants of weight two, and note that H does not vanish since $R = H^2 - 4\Delta\varphi\Delta\psi \geqq 0$. Now if φ_{yy} and ψ_{yy} are both different from zero it has been proved that the sign of U_ϕ is the same as that of

$$\varphi_{yy}\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_2)\},$$

and $U_\phi \neq 0$. Evidently if one but not both of the derivatives, φ_{yy} and ψ_{yy}, vanishes $U_\phi \neq 0$. If also φ_{xx} and ψ_{xx} are both different from zero the sign of

$$U_\phi' = \psi_{xx}H + 2\varphi_{xx}\Delta\psi$$

is the same as that of

$$\varphi_{xx}\left\{\left(\frac{1}{\alpha_1} - \frac{1}{\beta_1}\right)\left(\frac{1}{\alpha_2} - \frac{1}{\beta_1}\right) + \left(\frac{1}{\alpha_1} - \frac{1}{\beta_2}\right)\left(\frac{1}{\alpha_2} - \frac{1}{\beta_2}\right)\right\}$$
$$= \varphi_{xx}\frac{\varphi_{yy}}{\varphi_{xx}}\left\{\frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)}{\beta_1^2} + \frac{(\alpha_1 - \beta_2)(\alpha_2 - \beta_2)}{\beta_2^2}\right\},$$

and consequently in the cases before us, $R \geqq 0$, the same as that of U_ϕ. Any given non-singular transformation,

$$p = ap' + bq', \qquad q = cp' + dq',$$

* Bôcher, l. c., pp. 166, 236.

may be regarded as the transformation, $\theta = 1$, of the transformations θ,

$$p = a(\theta)p' + b(\theta)q', \qquad q = c(\theta)p' + d(\theta)q',$$

where the four functions of θ, a, b, c, d, are single-valued functions, continuous for $0 \leqq \theta \leqq 1$, and subject to the conditions:

$$a(0) = 1, \quad a(1) = a; \qquad b(0) = 0, \quad b(1) = b; \qquad c(0) = 0, \quad c(1) = c;$$
$$d(0) = 1, \quad d(1) = d.$$

Evidently $\theta = 0$ gives the identical transformation. If for the given transformation $ad - bc$ is positive the four functions of θ may be chosen so that in the interval named

$$a(\theta)d(\theta) - b(\theta)c(\theta)$$

does not vanish, and consequently no transformation θ is singular. For the same interval U_ϕ is a continuous function of θ. If $R > 0$ not both φ_{yy} and ψ_{yy} vanish for any θ, and U_ϕ has the same sign for $\theta = 0$ and $\theta = 1$. If for the given transformation $ad - bc$ is negative let us suppose, as we may without loss of generality, that originally no one of the four derivatives φ_{xx}, φ_{yy}, ψ_{xx}, ψ_{yy} is zero. Let the transformation with $\delta = -1$,

$$x' = y, \qquad y' = x,$$

be first applied; U_ϕ is transformed without change of sign to U_ϕ', and may then be proved, as in the case of positive determinant, to be unchanged in sign by the given transformation.

The case $R = 0$, $\alpha_1 = \beta_1$, $\alpha_2 \neq \beta_2$, needs further consideration, for $U_\phi = 0$ if $\varphi_{yy} = \psi_{yy} = 0$. If however $U_\phi = 0$ necessarily $U_\phi' \neq 0$, for the possibility,

$$\varphi_{xx} = \varphi_{yy} = \psi_{xx} = \psi_{yy} = 0,$$

is excluded. It remains to show that U_ϕ and U_ϕ' have the same sign when neither is zero. Suppose, as before, that for $\theta = 0$ none of the four derivatives vanishes; then U_ϕ and U_ϕ' have the same sign. When θ increases from 0 to 1 U_ϕ and U_ϕ' have continually the same sign unless one, say U_ϕ, vanishes for $\theta = \theta_1$, when we must have $\varphi_{yy} = \psi_{yy} = 0$. It is now conceivable that, for further increase of θ, U_ϕ should change sign if either φ_{yy} or ψ_{yy} remains zero while the other is not zero. We prove that, with our assumptions, this cannot occur. Suppose, for $\theta = \theta_1$, $\varphi_{yy} = \psi_{yy} = 0$; that, for all values of θ such that $\theta - \theta_1$ is positive and less than some positive number ϵ, $\varphi_{yy} = 0$, $\psi_{yy} \neq 0$. From $R = 0$, we have

$$\varphi_{xx}^2\psi_{yy} - 4\varphi_{xx}\varphi_{xy}\psi_{xy} + 4\varphi_{xy}^2\psi_{xx} = 0,$$

for any value of θ between θ_1 and $\theta_1 + \epsilon$. This equation is satisfied by 1. $\varphi_{xx} = \psi_{xx} = 0$. This is inadmissible, since we cannot have, for θ_1, $\varphi_{xx} = \varphi_{yy} = \psi_{xx} = \psi_{yy} = 0$; 2. $\varphi_{xx} = \varphi_{xy} = 0$. This case is also excluded, for we cannot have, at θ_1, $\varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0$; 3. if $\varphi_{xx} \neq 0$,

$$\psi_{yy} = \frac{4\varphi_{xy}}{\varphi_{xx}^2}(\varphi_{xx}\psi_{xy} - \varphi_{xy}\psi_{xx}), \qquad \epsilon > \theta - \theta_1 > 0.$$

Since all the derivatives are continuous functions of θ and since, for θ_1, $\psi_{yy} = 0$ we should have in this case for θ_1, and consequently for all transformations, again the excluded case

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

Consequently our hypothesis is inadmissible, and U_Φ is always of the same sign when it is different from zero. In the case III, *b*, 2, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0, \Delta\varphi = \Delta\psi = 0$; $H \neq 0$, we have $U_\Phi = \psi_{yy}H$. It has been proved that ψ_{yy} cannot change sign though it may vanish, but not both ψ_{yy} and ψ_{xx} vanish. In *b*, 3, $\Delta\varphi = 0$, $\Delta\psi > 0$, we have, if $H = 0$, $U_\Phi = \varphi_{yy}\Delta\psi$ and, since φ_{yy} may vanish but cannot change sign, the same is true of U_φ; not both φ_{yy} and φ_{xx} vanish. If $H \neq 0$ the proof given above for *c*, 3, $R > 0$, applies, and U_φ can neither vanish nor change sign. For *c*, 2, $\Delta\varphi > 0$, $\Delta\psi = 0$; $H \neq 0$, we have, as in *b*, 2, $U_\Phi = \psi_{yy}H$. If $H = 0$ the discussion fails.

We remark that our discussion proves that U_Φ and U_ψ have the same or different signs when neither is zero in the cases considered as H is positive or negative.

It is of interest to note that our discussion proves the existence of certain absolute conditional invariants or conditional differential parameters, for example

$$\frac{D_\Phi\psi}{\Delta\varphi}, \qquad \frac{H}{\sqrt{\Delta\varphi\Delta\psi}}, \qquad \frac{\Delta\varphi}{\Delta\psi},$$

the first with the conditions, $\varphi_x = \varphi_y = 0$, the second and third with the conditions $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

§ 5. **Examples.**—We give in this section simple examples illustrating each case of the theory developed in the preceding sections, also examples showing, in each case where the discussion is said to fail, that there may exist a maximum or minimum or no extreme. We shall illustrate the reciprocity in problems A and B, and the invariance of the conditions in some of the examples by interchanging the variables x and y. All of the examples are so chosen that $x_0 = y_0 = u_0 = v_0 = 0$.

For the general case of § 1, φ_x and φ_y not both zero, ψ_x and ψ_y not both zero.

1. $$\varphi = y, \qquad \psi = y - x^2.$$

We have $\varphi_y = \psi_y = 1$; $J = 0$; $y_\phi'' = 0, y_\psi'' = 2$. The curves, $y = 0$ and $y = x^2$, are tangent at $(0, 0)$ but do not osculate. In A we have a relative minimum, in B a relative maximum; φ_y and ψ_y have the same sign.

2. $$\varphi = y, \qquad \psi = y - x^3, \quad y - x^4, \quad y + x^4.$$

For each ψ in this example the curves, $y = 0$ and $x = 0$, osculate at the origin; for the three choices φ has respectively no extreme, a minimum, a maximum.

For § 2, case I, φ_x and φ_y not both zero, $\psi_x = \psi_y = 0$.

a. $\Delta\psi < 0$.

3. $$\varphi = y, \qquad \psi = x^2 + y^2.$$

Evidently no real branch of C, $\psi = 0$, passes through the origin.

b. $\Delta\psi = 0$. The point P (0, 0) is a cusp, an osculating point, or an isolated point of C.

4. $$\varphi = y, \qquad \psi = x^2 - y^3, x^2 + y^3, x^2 + y^4.$$

For the first two ψ, the origin is a cusp of C, and $D_\psi\varphi = 2$. For the first ψ there is a minimum, for the second a maximum of φ. For the third choice of ψ the origin is an isolated point of C and φ has no relative extreme.

5. $$\varphi = y, \qquad \psi = x^3 - y^2, \quad (y - x^2)^2 - x^5, \quad (y + x^2)^2 - x^5.$$

For each choice of ψ the origin is a cusp of C, and $D_\psi\varphi = 0$. For the three choices φ has respectively no extreme, a minimum, a maximum.

6. $$\varphi = y, \qquad \psi = y^2 - x^4 + x^5.$$

The origin is an osculating point of C, and $D_\psi\varphi = 0$. For a path on C with continuously turning tangent φ has at the origin no extreme, a minimum, or a maximum, depending on the path.

c. $\Delta\psi > 0$.

7. $$\varphi = y, \qquad \psi = xy + x^4, \quad xy - x^3, \quad xy + x^3.$$

For each value of ψ the origin is a double point of C with two distinct tangents, and $D_\psi\varphi = 0$. If it be desired that C be irreducible y^2 may be added to each value of ψ without affecting the results. For the three choices of ψ the value zero is respectively no extreme, a minimum, a maximum of φ for that branch of C whose tangent is not parallel to the horizontal tangent to S at Q.

For § 2, case II, $\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero.

a. $\Delta\varphi < 0$.

8. $$\varphi = x^2 + y^2, \quad - x^2 - y^2, \qquad \psi = y.$$

The two choices of φ give $D_\phi\psi$ the values 2 and -2 respectively. For the first φ has a minimum, for the second a maximum.

b. $\Delta\varphi = 0$.

9. $$\varphi = x^2 - y^3, \quad x^2 + y^3, \qquad \psi = y.$$

Both choices of φ give $D_\phi\psi = 2$, and in both cases φ has a minimum. It is evident on comparing this example with example 4 that we have no basis for comparing the extremes of problems A and B with the hypotheses of I*b*. Similarly if we exchange φ and ψ as given in examples 5 and 6 we have for each $D_\phi\psi = 0$, from 5 no extreme and two minima, from 6 a maximum.

c. $\Delta\varphi > 0$.

10. $$\varphi = x^2 - y^2, \qquad \psi = x, y.$$

The two values of ψ give $D_\phi\psi$ equal to -2 and 2 respectively and give φ maximum and minimum values respectively. For problem B, corresponding to I*c* there is no extreme. If we interchange φ and ψ as given in example 7 we have D_φ and $\psi = 0$ and have a maximum in the first case, no extreme in the second and third. There is evidently no possibility of comparing the nature of the extremes in A and B in this case furnished by our discussion.

For § 2, case III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

a. $\Delta\varphi < 0$. 1. $\Delta\psi < 0$.

11. $$\varphi = x^2 + y^2, \qquad \psi = 2x^2 + y^2.$$

There is no relative extreme in either A or B.

2. $\Delta\psi = 0$.

12. $$\varphi = x^2 + y^2.$$

Let ψ be given any of the values assigned in examples 4, 5, 6. For all choices φ has a minimum and $\varphi_{yy} = \varphi_{xx} = 2$.

3. $\Delta\psi > 0$.

13. $$\varphi = x^2 + y^2.$$

Let ψ have the values of 7. There is a minimum of φ for each branch of C.

b. $\Delta\varphi = 0$. 1. $\Delta\psi < 0$.

14. $$\varphi = y^2, \qquad \psi = x^2 + y^2.$$

Clearly φ has no relative extreme at the origin.

2. $\Delta\psi = 0$.

15. $$\varphi = y^2, \qquad \psi = x^2.$$

We find $\varphi_{yy} = 2$, $H = 4$, $U_\phi = 0$, $U_\phi' = 8$, $U_\psi = 8$, $U_\psi' = 0$.

Problems A and B have like extremes, both minima.

16. $$\varphi = -y^2, \qquad \psi = x^2.$$

We have

$$\varphi_{yy} = -2, \quad H = -4, \quad U_\phi = 0, \quad U_\phi' = -8, \quad U_\psi = 8, \quad U_\psi' = 0.$$

Problems A and B have unlike extremes, a maximum in A, a minimum in B.

The case, $\Delta\varphi = \Delta\psi = R = H = 0$, is excluded, since, as previously stated, we have, when these all vanish,

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

The following example shows that in this case φ may have no extreme, a minimum or a maximum:

17. $$\varphi = y^2, -y^2, \qquad \psi = y^2 - x^3, \quad y^2 - x^4,$$

3. $\Delta\psi > 0$.

18. $$\varphi = y^2, \qquad \psi = x^2 - y^2.$$

We find

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = -2, & \Delta\psi = 4, & H = 4, \\ U_\phi = 8, & U_\phi' = 8, & U_\psi = 8, & U_\psi' = 0. \end{array}$$

Problems A and B have like extremes, both minima.

19. $$\varphi = y^2, \qquad \psi = y^2 - x^2.$$

We have

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = 2, & \Delta\psi = 4, & H = -4, \\ U_\phi = 8, & U_\phi' = 8, & U_\psi = -8, & U_\psi' = 0. \end{array}$$

Problems A and B have unlike extremes, minima in A, a maximum in B.

20. $$\varphi = y^2 + x^3, \quad y^2 + x^4, \quad y^2 - x^4, \qquad \psi = xy.$$

We have for all choices of φ

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = 0, & \Delta\psi = 1, & H = 0, \\ U_\phi = 4, & U_\phi' = 0, & U_\psi = 0, & U_\psi' = 0. \end{array}$$

Problem A has for the three choices of φ respectively a minimum and no extreme, two minima, a maximum and a minimum. In each case the nature of the known extreme, a minimum, is given by the positive sign of U_ϕ. The discussion fails for B.

c. $\Delta\varphi > 0$. 1. $\Delta\psi < 0$. There is no extreme in problem A.

2. $\Delta\psi = 0$.

21. $$\varphi = x^2 - y^2, \qquad \psi = y^2, \quad y^2 \pm x^3.$$

We find $H = 4$, $U_\phi = 8$, $U_\phi' = 0$. There are a minimum in problem A, two minima in B. See example 18.

22. $$\varphi = xy + x^3, \ xy + x^4, \ xy - x^4, \qquad \psi = y^2, \quad y^2 - x^8 + x^9.$$

We have $H = U_\phi = U_\phi' = 0$. For the three choices of φ problem A has, for a path C with continuously turning tangent, respectively no extreme, a minimum, and a maximum. The discussion fails to give any information. See the reference to problem B in example 20.

3. $\Delta\psi > 0$.

23. $$\varphi = x^2 - y^2, \qquad \psi = 2x^2 - 5xy + 2y^2.$$

We find $\Delta\varphi = 4$, $\Delta\psi = 9$, $H = 0$, $R = -144$. In problem A the values of φ for the two branches of C are $3y^2$ and $-3x^2$, a minimum and a maximum respectively. In B the two values of ψ are $9y^2$ and $-x^2$, a minimum and a maximum.

24. $$\varphi = x^2 - y^2, \qquad \psi = x^2 - 4y^2,$$

We find

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 16, & H = -20, & R = 144, \\ U_\phi = 96, & U_\phi' = 24, & U_\psi = -24, & U_\psi' = -24. \end{array}$$

In problem A there are two like extremes, since $R > 0$, minima since $U_\phi > 0$, in B two like extremes, maxima since $U_\psi < 0$, unlike those of A since $H < 0$.

25. $$\varphi = x^2 - y^2, \qquad \psi = 4y^2 - x^2.$$

We have

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 16, & H = 20, & R = 144, \\ U_\phi = 96, & U_\phi' = 24, & U_\psi = 24, & U_\psi' = 24. \end{array}$$

There are two like extremes, both minima, in both A and B, of the same kind since $H > 0$.

26. $$\varphi = x^2 - y^2 + x^3, \ \ x^2 - y^2 \pm x^4, \qquad \psi = xy - y^2.$$

We have

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 1, & H = -4, & R = 0, \\ & U_\phi = U_\phi' = 4, & U_\psi = U_\psi' = -8. & \end{array}$$

For that branch of C whose tangent is not parallel to an asymptotic tangent to S at Q, that is for $y = 0$, φ has a minimum value, since $U_\phi > 0$; for the other branch, $x - y = 0$, the three choices of φ give respectively no extreme, a minimum, and a maximum. In problem B the branch of the curve,

$\varphi = 0$, whose tangent, $x + y = 0$, is not parallel to an asymptotic tangent to the surface, $\psi = 0$, at the origin, gives ψ a maximum value, since $U_\psi < 0$, unlike the extreme in A since $H < 0$.

27. $$\varphi = x^2 - y^2, \qquad \psi = y^2 - xy.$$

We have

$$\Delta\varphi = 4, \qquad \Delta\psi = 1, \qquad H = 4, \qquad R = 0,$$
$$U_\phi = U_\phi' = 4, \qquad U_\psi = U_\psi' = 8.$$

In this case, since $H > 0$, the extremes, different in A and B in example 26, are the same, minima in both problems.

March 28, 1920.